# A very sketchy outline of some basic harmonic analysis (Fourier analysis)

This is some notes of very sketchy outline for basic harmonic analysis (Fourier analysis). In this, we will approach such topics from one specific point of view : approximations in Banach spaces.

This note closely follows the approach of the text "An Introduction to Harmonic Analysis" by Yitzhak Katznelson.

**Question:** Let $B$ be a Banach space and let $x \in B$. How can we approximate $x$ via $x = \lim x_n$, where each $x_n$ is in some $B_n \subset B$, and structure of $B_n$ is easier to understand.

In case $B = C[0,1]$, the following Taylor expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

are typical examples of approximating $e^x$ and $\sin x$ via polynomials.

### Approximations in Hilbert Spaces

In case $B$ is a Hilbert space. That is, $B$ is a Hilbert space $H$. We just need to find a orthonormal basis $\{e_i\}_{i\in\mathcal{A}}$ of $H$. Then we have

$$x = \sum_{i\in\mathcal{A}} \langle x, e_i\rangle e_i,$$

which is exactly an approximation of $x$ via sub-Hilbert spaces which are generated by subsets of $\{e_i\}_{i\in\mathcal{A}}$.

### Approximations in Banach Spaces

For general Banach space $B$, it is not that easy to do approximations just as the case above for Hilbert spaces. There are no such things as orthonormal basis and no such things as Parseval's theorem for Banach spaces.

Consider $L^1(\mathbb{T}, \mu)$, where the measure $\mu$ is the probability Lebesgue measure. That is, $\mu$ is translation invariant and $\mu(\mathbb{T}) = 1$. For simplicity of notation, we use $L^1(\mathbb{T})$ to denote $L^1(\mathbb{T}, \mu)$.

Easy to check that $C(\mathbb{T}) \subset L^1(\mathbb{T})$ and $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$ for all $p > 1$.

As will show later in this note, lots of results that hold for $L^1(\mathbb{T})$ also hold for $C(\mathbb{T})$ and $L^p(\mathbb{T})$ for $p > 1$.

According to the discussions above for Hilbert spaces, for any $f \in L^2(\mathbb{T})$, it follows that

$$f = \lim_{n\to\infty} \sum_{k=-n}^{n} \langle f, z^n\rangle z^n = \sum_{k=-n}^{n} \frac{1}{2\pi} \int_0^{2\pi} f(s) \cdot e^{-iks}\, \mathrm{d}s \cdot e^{ikt}$$

in $L^2(\mathbb{T})$.

However, for any $f \in L^1(\mathbb{T})$, we do not always have

$$f = \lim_{n\to\infty} \sum_{k=-n}^{n} \langle f, z^n\rangle z^n$$

in $L^1(\mathbb{T})$.

One approach to show this is to use the Uniform Boundedness Theorem. One good explanation of this kind can be found in Rudin book, as an application of the Uniform Boundedness Theorem.

**Remark:** As for $C(\mathbb{T})$, it is easy to see that it also can be identified as, for any given $L \in \mathbb{R}_{>0}$, $L$-periodic functions over $\mathbb{R}$.

**Fact:** Although we can not expect to have $f = \lim_{n\to\infty} \sum_{k=-n}^{n} \langle f, z^n\rangle z^n$ for $f \in L^1(\mathbb{T})$, we can fix this problem by modifying the way how sums are made. That is, for any given $f \in L^1(\mathbb{T})$, let

$$\sigma_0(f) = \langle f, 1\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,\mathrm{d}t,\ \sigma_1(f) = \sum_{k=-1}^{1} \langle f, z^k\rangle z^k,\ \sigma_2(f) = \sum_{k=-2}^{2} \langle f, z^k\rangle z^k, \cdots.$$

Although we do not surely have $\sigma_n(f) \to f$ in $L^1(\mathbb{T})$, we do have $S_n(f) \to f$ in $L^1(\mathbb{T})$, where

$$S_0(f) = \sigma_0(f),\ S_1(f) = \frac{\sigma_0(f) + \sigma_1(f)}{2}, \cdots,\ S_n(f) = \frac{\sum_{k=0}^{n} \sigma_k(f)}{n+1}, \cdots.$$

Those $\{S_n\}$s are just the algebraic average of the naive $\{\sigma_n\}$s, and are called Cesàro sums.

A bit of easy calculation will indicate that

$$S_n(f) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \langle f, z^k\rangle z^k.$$

For the rest of the notes, we will give a sketchy proof of the fact above.

**Definition 1.** *For $f \in L^1(\mathbb{T})$, we define $\widehat{f}(n)$, the $n$-th Fourier coefficient of $f$, to be*

$$\widehat{f}(n) = \langle f, z^n\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt.$$

**Proposition 2.** *Let $f, g \in L^1(\mathbb{T})$. Then*

*i) $\widehat{(f+g)}(n) = \widehat{f}(n) + \widehat{g}(n)$ ;*
*ii) $\widehat{\alpha f}(n) = \alpha\widehat{f}(n), \quad \forall \alpha \in \mathbb{C}$;*
*iii) $\widehat{\overline{f}}(n) = \overline{\widehat{f}(-n)}$ ;*
*iv) $\widehat{f_\tau}(n) = \widehat{f}(n) \cdot e^{-in\tau}$, where $f_\tau$ is defined as $f_\tau(t) = f(t-\tau)$.*
*v) $|\widehat{f}(n)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(t)|\,dt = \|f\|_{L^1(\mathbb{T})}$*

**Corollary 3.** *For $f \in L^1(\mathbb{T})$ and $\{f_k\} \subset L^1(\mathbb{T})$, if $f_k \to f$ in $L^1$-norm, then $\widehat{f_k}(n) \to \widehat{f}(n)$ uniformly (regardless of $n$).*

**Theorem 4.** *Let $f \in L^1(\mathbb{T})$. Assume $\widehat{f}(0) = 0$ and define*

$$F(t) = \int_0^t f(s)\,ds.$$

*Then $F$ is continuous, $2\pi$-periodic and $\widehat{F}(n) = \frac{1}{in}\widehat{f}(n)$.*

**Remark:** This theorem above indicates that, under mild conditions, the Fourier coefficient of the integrand is not quite different from the Fourier coefficient of the original function. It is then natural to see that, under mild conditions, the Fourier coefficients of the derivatives are not quite different from the Fourier coefficients of the original functions. That is one of the reasons why Fourier analysis is useful in solving differential equations.

**Definition 5.** *For $f, g \in L^1(\mathbb{T})$, we define $f * g$, the convolution of $f$ and $g$, as*

$$(f * g)(t) = \int_{\mathbb{T}} f(t-\tau)g(\tau)\, d\mu = \frac{1}{2\pi}\int_0^{2\pi} f(t-\tau)g(\tau)\, d\tau.$$

**Proposition 6.** *For $f, g \in L^1(\mathbb{T})$, $f * g$ is also in $L^1(\mathbb{T})$ and*

$$\|f * g\|_{L^1(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})} \cdot \|g\|_{L^1(\mathbb{T})}.$$

**Proposition 7.** *For $f, g \in L^1(\mathbb{T})$, we have*

$$\widehat{(f * g)}(n) = \widehat{f}(n) \cdot \widehat{g}(n).$$

**Proposition 8.** *For $f, g, h \in L^1(\mathbb{T})$, we have*
*i) $f * g = g * f$ ;*
*ii) $f * (g + h) = f * g + f * h$ ;*
*iii) $f * (g * h) = (f * g) * h$.*

**Lemma 9.** *Assume $f \in L^1(\mathbb{T})$ and let $\varphi(t) = e^{int}$ for certain $n \in \mathbb{Z}$. Then*

$$(f * \varphi)(t) = \widehat{f}(n) \cdot e^{int}.$$

**Remark:** Following the proof of the lemma above, we can easily check the following: Let $\varphi_n(t) = \sum_{k=-n}^{n}(1 - \frac{|k|}{n+1})e^{int}$. Then, for any $f \in L^1(\mathbb{T})$,

$$S_n(f) = \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)\langle f, z^k\rangle z^k = f * \varphi_n.$$

In order to show that $S_n$ approaches $f$ with respect to $L^1$ norm, we just need to show that $f * \varphi_n$, the convolution of $f$ and $\varphi_n$, is approaching $f$ as $n \to \infty$. This is not totally new for us, as we have done similar things already in previous lectures on mollifiers (just recall how we can show that $C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ using mollifiers).

**Definition 10.** *A summable kernel is a sequence $\{K_n\}$ of continuous $2\pi$-periodic functions s.t.*
*1) $\frac{1}{2\pi}\int_0^{2\pi} K_n(t)\, dt = 1$ ;*
*2) $\frac{1}{2\pi}\int_0^{2\pi} |K_n(t)|\, dt < \infty$ ;*
*3) For any delta with $0 < \delta < \pi$, $\lim_{n\to\infty}\int_\delta^{2\pi-\delta} |K_n(t)|\, dt = 0$.*
*If all the $K_n$ are positive functions, then we call this $\{K_n\}$ a positive summable kernel.*

**Lemma 11.** *Let $B$ be a Banach space, and let $\varphi$ be a continuous $B$-valued functions on $\mathbb{T}$. Assume that $K_n$ is a summable kernel. Then*

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} K_n(t)\varphi(t)\, dt = \varphi(0).$$

**Proposition 12.** *Let $f \in L^1(\mathbb{T}, \mu)$, where $\mu$ is the Lebesgue measure. For any $\tau \in [0, 2\pi]$, define $f_\tau$ as $f_\tau(t) = f(t - \tau)$. Then for any $\tau_s \to \tau$, we have $f_{\tau_s} \to f_\tau$ in $L^1(\mathbb{T}, \mu)$.*

**Remark:** In the proposition above, if $\mu$ is no longer the Lebesgue measure, then the result might not hold. For example, consider the case $\mu$ is a point mass measure that is concentrated on one single point only.

**Lemma 13.** *Let $f \in L^1(\mathbb{T})$ and let $\{K_n\}$ be a summable kernel. Then*

$$f = \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} K_n(\tau) f_\tau \, d\tau \quad \text{in} \quad L^1(\mathbb{T}),$$

*where $f_\tau$ is defined as $f_\tau(t) = f(t - \tau)$.*

In this lemma, note that the right hand side of the equation $f = \cdots$ is the Riemann integration of a $L^1(\mathbb{T})$-valued continuous function $K_n(\tau)f_\tau$. Note that the continuity of $K_n$ follows from the definition of summable kernel and the continuity of $f_\tau$ follows from the proposition above.

**Lemma 14.** *Let $K\colon \mathbb{T} \to \mathbb{R}$ be continuous and let $f \in L^1(\mathbb{T})$. Then*

$$\frac{1}{2\pi} \int K(\tau) \cdot f_\tau \, d\tau = f * K,$$

*where $f_\tau$ is defined as above.*

In the lemma above ,note that the left hand side of the equation above is the Riemann integration of a $L^1(\mathbb{T})$-valued continuous function $K(\tau) \cdot f_\tau$. In contrast, the right hand side is a function whose value on each point $t$ is defined as $(f * K)(t) = \frac{1}{2\pi} \int_0^{2\pi} f(t - \tau) K(\tau) \, \mathrm{d}\tau$. One of the results above ensures that $\|f * K\|_{L^1(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})} \cdot \|K\|_{L^1(\mathbb{T})}$, which then ensures that this pointwisely defined function $f * K$ is in $L^1(\mathbb{T})$.

Based on those lemmas, we immediately have the following theorem.

**Theorem 15.** *Let $\{K_n\}$ be a summable kernel. Then for any $f \in L^1(\mathbb{T})$, we have*

$$f = \lim_{n\to\infty} f * K_n \quad \text{in} \quad L^1(\mathbb{T}).$$

Following the proof of the theorem above, we have:

**Corollary 16.** *Let $\{K_n\}$ be a summable kernel. Let a Banach space $B$ be $L^p(\mathbb{T})$ with $p > 1$ or let $B$ be $C(\mathbb{T})$. Then for any $f \in B$, we have*

$$f = \lim_{n\to\infty} f * K_n \quad \text{in} \quad B.$$

**Definition 17.** *$\{K_n\}$ with $K_n = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt}$ is called the Fejér kernel.*

Some easy calculations should lead you to the following proposition.

**Proposition 18.** *The Fejér kernel is a positive summable kernel.*

So far, we have proved $S_n(f) \to f$. That is, we have the following.

**Proposition 19.** *Let $\{K_n\}$ be a summable kernel. Let a Banach space $B$ be $L^p(\mathbb{T})$ with $p > 1$ or let $B$ be $C(\mathbb{T})$. For any $f \in B$, we have*

$$f = \lim_{n\to\infty} f * K_n = \lim_{n\to\infty} S_n(f) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \langle f, z^k\rangle z^k.$$

Based on this result, we can give an elegant proof of the Riemann-Lebesgue Lemma.

**Riemann-Lebesgue Lemma:** Let $f \in L^1(\mathbb{T})$. Then

$$\lim_{|n|\to\infty} \widehat{f}(n) \to 0.$$

*Proof.* Let $\{K_n\}$ be the Fejér kernel. Then $\|f - f * K_n\|_{L^1(\mathbb{T})} \to 0$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$\|f - f * K_N\|_{L^1(\mathbb{T})} < \epsilon.$$

Note that for any $n$ with $|n| > N$, $\widehat{f * K_N}(n) = 0$. It then follows that, for any $n$ with $|n| > N$,

$$\widehat{f}(n) = \widehat{f}(n) - 0 = \widehat{f}(n) - \widehat{f * K_N}(n) = \widehat{(f - f * K_N)}(n).$$

It then follows that

$$|\widehat{f}(n)| = |\widehat{(f - f * K_N)}(n)| \leq \|f - f * K_N\|_{L^1(\mathbb{T})} < \epsilon$$

for all $n$ with $|n| > N$. $\square$

**Remark:** As $e^{int} = \cos(nt) + i\sin(nt)$, and noting the Problem 10 of HW 4 for last term ("Real and Complex Analysis (I)"), which states that for any bounded measurable function $f$ on $[0, 1]$, we have $\int_0^1 f(x)\sin(nx)\,\mathrm{d}x \to 0$ as $n \to \infty$, you should be able to figure out, noting that any $L^1$ function on a finite measure space can be approximated by bounded simple functions, a brute force proof of the Riemann-Lebesgue Lemma above.